Viability and Arbitrage under Knightian Uncertainty

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Outline

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The Relation of Economics and Finance

- Mathematical Finance:
  - Take a probabilistic model of asset prices as given
  - $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ filtered probability space, (S_t^d) adapted processes
  - impose no arbitrage
  - develop a theory of derivative prices
- Economics:
  - asset prices are endogenous objects
  - derived by demand and supply on a competitive market

Given an arbitrage–free finance model (S_t^d), can one construct a plausible economy such that (S_t^d) are equilibrium prices?

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- After Hobson 1998, many papers on robust hedging, optimal transport, duality
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Viability and Arbitrage under Risk

Harrison, Kreps, Martingales and arbitrage in multiperiod securities markets, Journal of Economic Theory, 1979

Framework: Filtered Probability Space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$

- space of contingent claims is $L^2(P)$
- P fixes the notion of "similar commodities", i.e. the topology
- and the notion of "negligible event", here: P-null sets
- and the notion of order, here P-a.s. greater or equal

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Conceivable Agents

- $\mathcal{A}$ is the set of preferences $\preceq$ (complete, transitive orderings) on $\mathcal{X} = L^2(P)$ satisfying convexity, continuity, and strict monotonicity:
- for all $X \in \mathcal{X}$, the upper contour set $\{Z \in \mathcal{X} : X \preceq Z\}$ is convex,
- for all $X \in \mathcal{X}$, the upper contour set $\{Z \in \mathcal{X} : X \preceq Z\}$ is closed under L^2–convergence,
- if $P[R \geq 0] = 1$ and $P[R > 0] > 0$, then $X \prec X + R$ for all $X \in \mathcal{X}$

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Financial Market

- let $(\mathcal{F}_t)_{t=0,\dots,T}$ be a filtration with $\mathcal{F}_0$ trivial, $\mathcal{F}_T \subseteq \mathcal{F}$
- let $S_t^0 = 1$ be a numéraire,
- for $d = 1, \dots, D$, let $S^d = (S_t^d)_{t=0,\dots,T}$ be adapted, positive asset prices
- gains from trade for a self–financing portfolio $\theta = (\theta_t)$

$$G^\theta = \sum_{t=1}^{T} \theta_t \cdot \Delta S_t$$

- θ is an arbitrage if $P[G^\theta \geq 0] = 1$ and $P[G^\theta > 0] > 0$

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Viability and Arbitrage under Risk

Representative Agent Equilibrium

- The utility maximization problem for a conceivable agent $\preceq$ is well–posed (at 0) if for every self–financing portfolio θ we have $G^\theta \preceq 0$
- The financial market S is viable if for some conceivable agent $\preceq \in \mathcal{A}$, the utility maximization problem is well–posed at 0.
- (Then the market consisting of this "representative agent" is in equilibrium.)

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Viability and Arbitrage under Risk

Theorem (Harrison, Kreps 1979)

The financial market S is viable if and only if there is no arbitrage.

Viability and Arbitrage under Risk

Viability implies no arbitrage

The strict upper contour set at 0 is convex and open in $L^2(P)$ and disjoint from all gains from trade. By the separation theorem, there exists a L^2–continuous, P-strictly positive linear functional that separates the sets. This allows to define an "equivalent martingale measure", hence no arbitrage.

Viability and Arbitrage under Risk

No arbitrage implies viability

- Modern version: by Dalang, Morton, Willinger 1990, FTAP, there exists an equivalent martingale measure P^*.
- Define a linear preference relation $\preceq$ by

$$X \preceq Y \text{ iff } E^{P^*}X \leq E^{P^*}Y$$

- $\preceq$ bis $L^2(P)$–continuous and P–strictly increasing (equivalence!)
- and the utility maximization problem is well–posed at 0.

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Knightian Uncertainty - Issues

Consider the robust model in which uncertainty is described by a non–dominated class of probability measures $\mathcal{P}$.

- Typical utility function (Gilboa, Schmeidler): $U(X) = \inf_{P\in\mathcal{P}} E^P u(X)$
- θ arbitrage (Vorbrink 2014, Bouchard, Nutz 2015) if
  - $P[G^\theta \geq 0] = 1$ $\mathcal{P}$–quasi surely
  - for some $P \in \mathcal{P}$, $P[G^\theta > 0] > 0$

Knightian Uncertainty - Issues

- Then the utility maximization can be well posed at 0 even if there is arbitrage, $U(G^\theta) = U(0)$
- because $P_0[G^\theta = 0] = 1$ for the worst–case measure P_0
- the set of strictly monotone, convex preferences is empty
- there do not exist strictly positive linear functionals (compare Beissner, Denis, 2018)

Knightian Uncertainty - Issues

- there is no hope to construct a representative agent equilibrium supporting an arbitrage–free market
- can there be arbitrage in equilibrium?

Knightian Uncertainty - New Approach

The common ordering

- $(\Omega, \mathcal{F})$ measurable space
- $(\mathcal{H}, \tau, \leq)$ pre–ordered topological vector space of measurable functions containing the constants
- $Z \in \mathcal{H}$ is negligible if $Z \leq 0$ and $Z \geq 0$

Marketed Space

The zero cost trades are given by a convex cone $\mathcal{I}$

1. Usually, the set $\mathcal{I}$ consists of (suitably restricted) stochastic integrals
2. of the form $G^\theta = \sum_{t=1}^{T} \theta_t \cdot \Delta S_t$ in discrete time
3. In Harrison–Kreps, the market is described by a marketed space $M \subset L^2(\Omega, \mathcal{F}, P)$ and a (continuous) linear functional π on M. In this case, $\mathcal{I}$ is the kernel of the price system, i.e.

$$\mathcal{I} = \{X \in M : \pi(X) = 0\}\,.$$

Relevant Contracts

A non–empty, convex set $\mathcal{R}$ of $\leq$–nonnegative payoffs describes the relevant contracts.
$\mathcal{R}$ contains all strictly positive constant contracts

Relevant Contracts

Examples

- probabilistic model: $\mathcal{R}$ contains the non-zero a.s. nonnegative random variables

$$P[X \geq 0] = 1, P[X > 0] > 0$$

- multiple prior uncertainty: $\mathcal{R}$ contains the non-zero q.s. nonnegative random variables

$$P[X \geq 0] = 1 \text{ for all } P \in \mathcal{P}$$
$$P[X > 0] > 0 \text{ for some } P \in \mathcal{P}$$

- $\mathcal{R} = (0, \infty)$

Agents

The set $\mathcal{A}$ of conceivable agents consists of all preference relations on $\mathcal{H}$ that are

- weakly monotone with respect to the order $\leq$
- convex
- τ–lower semicontinuous: for every sequence $X_n \to X$ with $X_n \preceq Y$ for all $n \in \mathbb{N}$, we have $X \preceq Y$

Viability

Definition

A financial market $(\mathcal{H}, \tau, \leq, \mathcal{I}, \mathcal{R})$ is *viable* if there exists a family of agents $\{\preceq_a\}_{a\in A} \subset \mathcal{A}$ and net trades $(\ell_a^*)_{a\in A} \subset \mathcal{I}$ such that

- l_a^* is optimal for each agent $a \in A$, i.e.

$$\forall a \in A,\ \ell \in \mathcal{I} \quad \ell \preceq_a \ell_a^*, \tag{1}$$

- the market clears, i.e. $\sum_{a\in A} l_a^* = 0$,
- for every relevant contract $R \in \mathcal{R}$ there exists an agent $a \in A$ such that $\ell_a^* \prec_a \ell_a^* + R$

Remarks

The market needs to see relevant contracts

- new property was free in probabilistic setting
- equivalent martingale measures “see” every non-zero positive random variable

Arbitrage

Definition

1. $l^* \in \mathcal{I}$ is an arbitrage if there exists $R \in \mathcal{R}$ with $l \geq R$
2. A sequence $(l_n) \subset \mathcal{I}$ is a free lunch with vanishing risk if there exist a sequence $c_n \downarrow 0$ and $R \in \mathcal{R}$ such that $c_n + l_n \geq R$.

Equivalence

Theorem

A financial market is viable if and only if there is no arbitrage.

The proof is based on strictly positive sublinear instead of linear preferences.

Sublinear Martingale Expectations

Definition

A functional $\mathcal{E} : \mathcal{H} \to \mathbb{R} \cup \{-\infty, \infty\}$ is a sublinear expectation if it is $\leq$-monotone, cash–additive, and sublinear.

$\mathcal{E}$

- is *absolutely continuous*, if $\mathcal{E}(Z) = 0$ for every negligible Z .
- has *full support* if $\mathcal{E}(R) > 0$, for every $R \in \mathcal{R}$.
- has the *(super–)martingale property* if $\mathcal{E}(\ell) \leq 0$ for every $\ell \in \mathcal{I}$.

Fundamental Theorem of Asset Pricing

The viability theorem is closely connected to the fundamental theorem of asset pricing.

Theorem

A financial market is viable if and only if there exists a lower semicontinuous sublinear martingale expectation with full support.

The sublinear martingale expectation is able to "see" all relevant contracts in the case when no strictly positive linear functionals exist.

Proof Sketch

The proof is based on a close analysis of the superhedging functional

$$\mathcal{D}(X) := \inf\{c \in \mathbb{R} \ : \ \exists \{\ell^n\}_{n=1}^{\infty} \subset \mathcal{I}, \{e_n\}_{n=1}^{\infty} \subset \mathcal{H},\ e_n \xrightarrow{\mathcal{T}} 0 \text{ such that } \mathbf{c} + e_n + \ell^n \geq X\}\,.$$

and its convex dual

$$\mathcal{D}^*(\phi) = \sup_{X \in \mathcal{H}} \phi(Y) - \mathcal{D}(Y)$$

for $\phi \in \mathcal{H}'$, the topological dual space of $\mathcal{H}$

Proof Sketch

Lemma

$dom(\mathcal{D}^*)$ is given by

$$\begin{aligned} dom(\mathcal{D}^*) &= \{ \varphi \in \mathcal{H}'_+ \ : \ \mathcal{D}^*(\varphi) = 0 \} \\ &= \{ \varphi \in \mathcal{H}'_+ \ : \ \varphi(X) \leq \mathcal{D}(X), \ \ \forall \ X \in \mathcal{H} \} . \end{aligned}$$

In particular,

$$\mathcal{D}(X) = \sup_{\varphi \in dom(\mathcal{D}^*)} \varphi(X), \quad X \in \mathcal{H}. \tag{2}$$

Furthermore, there are free lunches with vanishing risk whenever $dom(\mathcal{D}^*)$ is empty.

Proof Sketch

Under no arbitrage,

- $dom(\mathcal{D}^*)$ gives the family of preferences needed to construct the market for viability
- $\mathcal{D}$ is the sublinear expectation with full support

Outline

The Efficient Market Hypothesis

Fama, 1970: If financial markets are “informationally efficient”, then the expected returns of all assets are equal.

Strong Form of the EMH

Example

Suppose that agents agree to order contracts by

$$X \preceq Y \iff E^P X \leq E^P Y$$

for some probability measure P (and $\mathcal{H} = L^1(\Omega, , \mathcal{F}, P)$).
Suppose that we have a finite-time discrete market without constraints.
Then the expectation under P is the only full support martingale expectation and the strong form of the EMH results.

Weak Form of the EMH

Example

Suppose that agents agree to order contracts by

$$X \preceq Y \iff P[X \geq Y] = 1$$

for some probability measure P (and $\mathcal{H} = L^1(\Omega, , \mathcal{F}, P)$).

$$\mathcal{R} = \left\{ R \in L^1(\Omega, \mathcal{R}, \mathbb{P})_+ \ : \ \mathbb{P}(R > 0) > 0 \right\}.$$

Then the maximal sublinear martingale expectation is the one generated by equivalent martingale measures P^*; thus, we obtain the probabilistic weak form of the EMH: expected returns are equal under some equivalent probability measure.

Smooth Ambiguity Aversion

Example

Let $\mathcal{P}$ be a set of priors on $(\Omega, \mathcal{F})$.
Suppose that all conceivable agents satisfy the smooth ambiguity model of Klibanoff, Marinacci, Mukerji, 2005.
Then there exists a second–order prior μ on $\mathcal{P}$ and

$$X \preceq Y \iff \mu\left[P \in \mathcal{P} : P[X \geq Y] = 1\right] = 1.$$

Let the relevant contracts be the positive ones according to μ.

Smooth Ambiguity Aversion

Example

"second–order equivalent martingale measures" of the form

$$\mathbb{E}X = \int_{\mathcal{P}} E^P[DX]\mu(dP)$$

for some state–price density $D \geq 0$

Multiple Priors

Example

Let $\mathcal{P}$ be a set of priors on $(\Omega, \mathcal{F})$.

$$X \leq Y \iff X \leq Y \, \mathcal{P} - q.s.$$

$$\mathcal{R} = \{ R \in \mathcal{P} \; : \; \exists \, \mathbb{P} \in \mathcal{M} \quad \text{such that} \quad \mathbb{P}(R > 0) > 0 \, \} \, .$$

maximal sublinear martingale expectation is generated by a the set of probability measures $\mathcal{Q} = \mathcal{Q}(\Theta)$ which is equivalent to $\mathcal{M}$ such that all traded contracts are symmetric martingales under the sublinear expectation induced by $\mathcal{Q}$.

Conclusion

- Viability and Arbitrage in an order–theoretic framework under Knightian uncertainty
- Equivalence of absence of arbitrage and viability
- abstract version of the FTAP
- Old and new forms of EMH can be subsumed under this framework